Continuity and Differentiability

1. Are the following two functions $\dfrac{x^2-25}{x-5}$ and $x + 5$ identical?

If not, give reasons.

What value must be assigned to the discontinuity at $x = 5$ in order to remove the point of discontinuity at $x = 5$?

2. Show that $\dfrac{5}{x-1}+\dfrac{7}{x-2}+\dfrac{16}{x-3}=0$ has a solution between 1 and 2 , and another between 2 and 3 .

3. Prove that the function: $f(x)=\begin{cases}\dfrac{x^2}{k}-k & ,\text{for } 0<x<k\\ 0 & ,\text{for } x=k\\ k-\dfrac{k^3}{x^2} & ,\text{for } x>k\end{cases}$ is continuous at $x = k$.

4. Let $f(x)=\begin{cases}0 & ,\text{where } x=0\\ 1/2-x & ,\text{where } 0<x<1/2\\ 1/2 & ,\text{where } x=1/2\\ 3/2-x & ,\text{where } 1/2<x<1\\ 1 & ,\text{where } x=1\end{cases}$

Show that $f(x)$ assumes every value between 0 and 1 once and only once as x increases from 0 to 1 .

Find the discontinuities of $f(x)$.

5. In the following, sketch the graph of $f(x)$. For what value of a is f continuous? differentiable ?

(a) $f(x)=\begin{cases}x+1 & ,x>0\\ a & ,x=0\\ x-1 & ,x<0\end{cases}$ **(b)** $f(x)=\begin{cases}x & ,x\neq 0\\ a & ,x=0\end{cases}$

6. Test the continuity of $f(x)$ at $x = 0$ where

$$f(x)=\begin{cases}\sin x\cos\dfrac{1}{x} & ,\text{where } x\neq 0\\ 0 & ,\text{where } x=0\end{cases}$$

7. Let f be a continuous function from $[a, b]$ to $[a, b]$.

Show that there is always a x_0 in $[a,b]$ with $f(x_0) = x_0$.

8. The function f is defined by $f(x)=\begin{cases}\sin x & \text{for } x\leq 0\\ x & \text{for } x>0\end{cases}$

Sketch the graphs of $f(x)$ and its derivative $f'(x)$ for $-\pi<x<\pi$ and decide whether the functions f and f' are continuous at $x = 0$ or not.

9. Let $f(x)$, $g(x)$ be two continuous functions defined on $\mathbf{R}$ and $\phi(x) = \begin{cases} f(x) & \text{where } x^2 > 1 \\ g(x) & \text{where } x^2 < 1 \\ \frac{1}{2}[f(x)+g(x)] & \text{where } x = 1 \end{cases}$.

Show that $\phi(x) = \lim_{n\to\infty} \frac{x^n f(x) + g(x)}{x^n + 1}$, $x \neq 1$. Is $\phi(x)$ continuous on $\mathbf{R}$?

10. Show that $y = (\cos x)^{-1/2}$ is continuous in its domain, so is $y = \sec x + \csc x$.

11. Discuss the continuity of f :

(a) $f(x) = \begin{cases} \frac{x}{\sin x} & x \in (0, \pi/2] \\ 1 & x = 0 \end{cases}$ (at $x = 0$)

(b) $f(x) = x - [x]$ (at $x = 0$, $x = 1/2$)

(c) $f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$ (at $x = 0$, $x = \pi$)

12. **(a)** Sketch the graphs of the real-valued functions f_0, f_1 and f_2, where

$$f_0(x) = \begin{cases} \sin\frac{1}{x} & x \in \mathbf{R}, x \neq 0 \\ 0 & x = 0 \end{cases}, \quad f_1(x) = \begin{cases} x\sin\frac{1}{x} & x \in \mathbf{R}, x \neq 0 \\ 0 & x = 0 \end{cases}, \quad f_2(x) = \begin{cases} x^2\sin\frac{1}{x} & x \in \mathbf{R}, x \neq 0 \\ 0 & x = 0 \end{cases}$$

(b) With the aid of your sketches, explain why f_0 is not continuous at $x = 0$, but f_1 and f_2 are continuous at $x = 0$.

(c) A function is said to be differentiable at $x = a$ if $\lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$ exists. Use this definition to show that f_2 is differentiable at $x = 0$. Write down an expression for $\frac{df_2}{dx}$ where $x \neq 0$.

(d) Hence show that although f_2 is differentiable everywhere, its derivative is not continuous at $x = 0$.

13. Let $f(x) = \begin{cases} x^3\sin\frac{1}{x} & x \neq 0, \quad x \in \mathbf{R} \\ 0 & x = 0 \end{cases}$

Prove that:

(a) $f(x)$ is continuous at $x = 0$;

(b) $f(x)$ has a derivative at $x = 0$;

(c) $f'(x)$ is continuous at $x = 0$.